6.S966: Practice Exam 1, Spring 2024

Solutions

- This is a closed book exam. One page (8 1/2 in. by 11 in) of notes, front and back, are permitted. Calculators are not permitted.
- The total exam time is 1 hours and 20 minutes.
- The problems are not necessarily in any order of difficulty.
- Record all your answers in the places provided. If you run out of room for an answer, continue on a blank page and mark it clearly.
- If a question seems vague or under-specified to you, make an assumption, write it down, and solve the problem given your assumption.
- If you absolutely *have* to ask a question, come to the front.
- **Write your name on every piece of paper.**

Name: ______________________ MIT Email: ______________________

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

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Sudoku for Group Theorists

1. (20 points) Complete the following multiplication and character tables below using the Rearrangement Theorem and the First and Second Wonderful Orthogonal Theorems for Character.

   (a) Complete the following multiplication table.

   Solution:

	0	1	2	3	4	5	6	7
0	1	4	7	6	2	3	0	5
1	4	2	5	0	7	6	1	3
2	7	5	6	4	3	1	2	0
3	6	0	4	5	1	7	3	2
4	2	7	3	1	5	0	4	6
5	3	6	1	7	0	2	5	4
6	0	1	2	3	4	5	6	7
7	5	3	0	2	6	4	7	1

   (b) Cyclic groups are only irreducible over complex numbers (rather than real numbers), but have all 1D irreps. Complete the table below for the cyclic group C_3 where A_1 is the trivial irrep and A_2 and A_3 are (complex) 1D representations. $\omega = e^{i2\pi/3}$ and $\omega^2 = e^{i4\pi/3} = e^{-i2\pi/3}$. Hint: $1 + \omega + \omega^2 = 0$.

   Solution:

	E	C_3	$(C_3)^2$
A_1	1	1	1
A_2	1	ω	ω^2
A_3	1	ω^2	ω

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Parsing Proofs

2. (20 points) In this problem, we will present a single step of some of the proofs shown in class, exercises, or notes and ask what properties of matrices or groups or lemmas or theorem, allows us to take this step.

   (a) Part 1 of Schur's lemma states that if a square matrix M commutes with all the elements of an irreducible representation, then it must be of the form λI for some λ.

   The proof follows the following steps:

   1. Every representation is similar to one with only hermetian matrices
   2. if M computes with a set Hermetian matrices, so do $M1 = M + M^*$ and $M2 = i(M - M^*)$
   3. Each of $M1$ and $M2$ is hermitian. So they must be diagnolizable
   4. If a Matrix Mi commutes with a diagonal matrix D. Then either Mi has block diagonal form, or D is constant
   5. Every matrix in an irrep must be full rank.

   i. Let f be any class function, and D be an irreducible representation. $(f(g) = f(h^{-1}gh) \forall g, h \in \mathcal{G})$. Show that $X = \sum_{g \in \mathcal{G}} f(g) D(g)$ is a constant matrix.

   > **Solution:** $D(h^{-1}) X D(h) = \sum_{g \in \mathcal{G}} f(g) D(h^{-1}gh) = \sum_{g \in \mathcal{G}} f(hgh^{-1}) D(g) = X$

   ii. Find the constant in the matrix above in terms of the character of the irreducible reprsentation and the function f

   > **Solution:**
   > Take trace of both sides, we get $\frac{<f,\chi>}{l} = \lambda$ (where l) is the dimension of the representation.

   iii. Argue if D is not reducible, then if we take a similarity transform giving it block diagonal form where each block is an irrep. Then each block of X would still be a constant matrix with the same constant you found above.

   > **Solution:**
   > Consider the sum that defines X one block at a time and apply the previous part

   iv. Argue that the characters of the irreducible representations in the regular representation must span the space of all class functions.
   Hint: consider the orthognal complement, and compute X in two ways.

   > **Solution:**
   > Consider any nonezero function in the orthognal complement of the space of the class functions spanned by irreducible characters. If we apply the previous results on it and the regular representation We see that X must end up being the 0 matrix (as it's similar to it). However, since no two elements of the regular representation overlap anywhere, and f is not 0. This can not be true.

(b) The Wonderful Orthogonal Theorem can be written in two ways.

$$\sum_R D^{(\Gamma_j)}_{\mu\nu}(R) D^{(\Gamma_{j'})}_{\nu'\mu'}(R^{-1}) = \frac{h}{l_j}\delta_{\Gamma_j\Gamma_{j'}}\delta_{\mu\mu'}\delta_{\nu\nu'} \tag{1}$$

$$\sum_R D^{(\Gamma_j)}_{\mu\nu}(R) [D^{(\Gamma_{j'})}_{\mu'\nu'}(R)]^* = \frac{h}{l_j}\delta_{\Gamma_j\Gamma_{j'}}\delta_{\mu\mu'}\delta_{\nu\nu'} \text{ if the representations are unitary.} \tag{2}$$

i. Use properties of unitary matrices to derive line (2) from line (1).

Solution: If the representations are unitary $U^{-1} = U^\dagger = [U^*]^T$, thus $D^{(\Gamma_{j'})}_{\nu'\mu'}(R^{-1}) = [D^{(\Gamma_{j'})}_{\nu'\mu'}(R)]^\dagger = [[D^{(\Gamma_{j'})}_{\nu'\mu'}(R)]^*]^T = [D^{(\Gamma_{j'})}_{\mu'\nu'}(R)]^*$. We can then plug into the first expression to arrive at the second.

ii. We then proceeded to equate

$$\sum_R\sum_\mu D^{(\Gamma_j)}_{\mu\nu}(R) D^{(\Gamma_{j'})}_{\nu'\mu}(R^{-1}) = \sum_R\sum_\mu D^{(\Gamma_j)}_{\nu'\mu}(R^{-1}) D^{(\Gamma_{j'})}_{\mu\nu}(R) \tag{3}$$

Why were we able to reorder the representations in line (3)?

Solution: Index notation specifies the entries of tensors, which are just numbers which commute.

(c) In our proof of Case 2 of the Wonderful Orthogonality Theorem ($l_j = l_{j'}$ and $\Gamma_j = \Gamma_{j'}$) we arrived at the following equation:

$$c''_{\nu\nu'}\delta_{\mu\mu'} = \sum_R D^{(\Gamma_{j'})}_{\mu\nu}(R) D^{(\Gamma_{j'})}_{\nu'\mu'}(R^{-1}), \text{ where } c''_{\nu\nu'} = \frac{c}{c'_{\nu\nu'}} \tag{4}$$

We then chose $\mu = \mu'$ and summed over μ to start solving for $c''_{\nu\nu'}$

$$c''_{\nu\nu'}\sum_\mu \delta_{\mu\mu} = c''_{\nu\nu'} l_{j'} = \sum_R\sum_\mu D^{(\Gamma_{j'})}_{\mu\nu}(R) D^{(\Gamma_{j'})}_{\nu'\mu}(R^{-1}) \tag{5}$$

where $l_{j'}$ is the dimension of the $\Gamma_{(j')}$ representation.

i. Why were we allowed to make this choice ($\mu = \mu'$) and perform this sum over μ?

Solution: The right hand side is an identity matrix multiplied by $c''_{\nu\nu'}$. Thus, the only non-trivial entries are when $\mu = \mu'$

ii. We then proceeded to equate

$$\sum_R\sum_\mu D^{(\Gamma_{j'})}_{\mu\nu}(R) D^{(\Gamma_{j'})}_{\nu'\mu}(R^{-1}) = \sum_R\sum_\mu D^{(\Gamma_{j'})}_{\nu'\mu}(R^{-1}) D^{(\Gamma_{j'})}_{\mu\nu}(R) \quad = \sum_R D^{(\Gamma_{j'})}_{\nu'\nu}(R^{-1}R) \tag{6}$$

Why were we able to reorder the representations in this expression?

Solution: Index notation specifies the entries of tensors, which are just numbers which commute.

(d) In class, we proved that group convolution with respect to group $\mathcal{G}$ is equivalent under it's action. In particular:

$$L_g(f) \star \psi = L_g(f \star \psi) \tag{7}$$

Where $L_g f(x, y) = (g^{-1} \circ f)(x, y) = f(g^{-1}x, g^{-1}y)$.

i. If the group $\mathcal{H} \subset \mathcal{G}$ is the group of symmetries of the image function $f(L_h(f) = f, \forall h \in \mathcal{H})$. Argue that the correlation (with respect to group $\mathcal{G}$) is invariant under H.

Solution:

$$L_h(f \star \psi) = L_h(f) \star \psi = f \star \psi$$

ii. In general, What subset of $\mathcal{G}$ is group convolution (with respect to $\mathcal{G}$) invariant under when the input image is symmetric under group $\mathcal{H}$? briefly explain your reasoning.

Solution:
We can apply the previous problem on $\mathcal{G} \cap \mathcal{H}$

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Isomorphisms of Multiplication Tables of Order 8

3. (20 points) Below are five group multiplication tables of groups of order 8. Here, we are using various symbols instead of numbers, so it is clear in your answers which table you are giving answers to.

	a	b	c	d	f	g	h	j
a	c	f	j	h	d	a	b	g
b	f	c	d	g	j	b	a	h
c	j	d	g	b	h	c	f	a
d	h	g	b	c	a	d	j	f
f	d	j	h	a	g	f	c	b
g	a	b	c	d	f	g	h	j
h	b	a	f	j	c	h	g	d
j	g	h	a	f	b	j	d	c

Table 1

	α	β	γ	δ	ϵ	ζ	η	θ
α	θ	η	ζ	ϵ	δ	γ	β	α
β	η	θ	ϵ	ζ	γ	δ	α	β
γ	ζ	δ	θ	β	η	α	ϵ	γ
δ	ϵ	γ	η	α	θ	β	ζ	δ
ϵ	δ	ζ	β	θ	α	η	γ	ϵ
ζ	γ	ϵ	α	η	β	θ	δ	ζ
η	β	α	δ	γ	ζ	ϵ	θ	η
θ	α	β	γ	δ	ϵ	ζ	η	θ

Table 2

	①	②	③	④	⑤	⑥	⑦	⑧
①	①	②	③	④	⑤	⑥	⑦	⑧
②	②	①	④	③	⑥	⑤	⑧	⑦
③	③	④	①	②	⑦	⑧	⑤	⑥
④	④	③	②	①	⑧	⑦	⑥	⑤
⑤	⑤	⑥	⑦	⑧	①	②	③	④
⑥	⑥	⑤	⑧	⑦	②	①	④	③
⑦	⑦	⑧	⑤	⑥	③	④	①	②
⑧	⑧	⑦	⑥	⑤	④	③	②	①

Table 3

	◯	□	△	⋄	⬠	⬡	⯃	⋆
◯	□	⬠	⋆	⯃	△	⋄	◯	⬡
□	⬠	△	⬡	◯	⋆	⯃	□	⋄
△	⋆	⬡	⯃	⬠	⋄	□	△	◯
⋄	⯃	◯	⬠	⬡	□	⋆	⋄	△
⬠	△	⋆	⋄	□	⬡	◯	⬠	⯃
⬡	⋄	⯃	□	⋆	◯	△	⬡	⬠
⯃	◯	□	△	⋄	⬠	⬡	⯃	⋆
⋆	⬡	⋄	◯	△	⯃	⬠	⋆	□

Table 3

	♈	♉	♊	♋	♌	♍	♎	♏
♈	♌	♈	♍	♉	♋	♎	♏	♊
♉	♈	♉	♊	♋	♌	♍	♎	♏
♊	♏	♊	♌	♍	♎	♈	♉	♋
♋	♉	♋	♏	♌	♈	♊	♍	♎
♌	♋	♌	♎	♈	♉	♏	♊	♍
♍	♊	♍	♋	♎	♏	♌	♈	♉
♎	♍	♎	♉	♏	♊	♋	♌	♈
♏	♎	♏	♈	♊	♍	♉	♋	♌

Table 5

Name: ______________________________

(a) Give the symbol that corresponds to the identity element for each table

> **Solution:** g, θ, ①, ⯃, ♉

(b) For each symbol, give it's inverse.

> **Solution:**
>
> $a^{-1} \to j$
> $b^{-1} \to d$
> $c^{-1} \to c$
> $d^{-1} \to b$
> $f^{-1} \to f$
> $g^{-1} \to g$
> $h^{-1} \to h$
> $j^{-1} \to a$
>
> $\alpha^{-1} \to \alpha$
> $\beta^{-1} \to \beta$
> $\gamma^{-1} \to \gamma$
> $\delta^{-1} \to \epsilon$
> $\epsilon^{-1} \to \delta$
> $\zeta^{-1} \to \zeta$
> $\eta^{-1} \to \eta$
> $\theta^{-1} \to \theta$
>
> ①$^{-1} \to$ ①
> ②$^{-1} \to$ ②
> ③$^{-1} \to$ ③
> ④$^{-1} \to$ ④
> ⑤$^{-1} \to$ ⑤
> ⑥$^{-1} \to$ ⑥
> ⑦$^{-1} \to$ ⑦
> ⑧$^{-1} \to$ ⑧
>
> ◯$^{-1} \to$ ⋄
> □$^{-1} \to$ ⬡
> △$^{-1} \to$ △
> ⋄$^{-1} \to$ ◯
> ⬠$^{-1} \to$ $\star$
> ⬡$^{-1} \to$ □
> ⯃$^{-1} \to$ ⯃
> $\star^{-1} \to$ ⬠
>
> ♈$^{-1} \to$ ♋
> ♉$^{-1} \to$ ♉
> ♊$^{-1} \to$ ♎
> ♋$^{-1} \to$ ♈
> ♌$^{-1} \to$ ♌
> ♍$^{-1} \to$ ♏
> ♎$^{-1} \to$ ♊
> ♏$^{-1} \to$ ♍

(c) What is the order of each element of a group? Reminder, the order of an element is how many times you must multiply the element by itself to return to the identity.

Solution:

$a \to 4$	$\alpha \to 2$
$b \to 4$	$\beta \to 2$
$c \to 2$	$\gamma \to 2$
$d \to 4$	$\delta \to 4$
$f \to 2$	$\epsilon \to 4$
$g \to 1$	$\zeta \to 2$
$h \to 2$	$\eta \to 2$
$j \to 4$	$\theta \to 1$
①$\to$ 1	$\bigcirc \to 8$
②$\to$ 2	$\square \to 4$
③$\to$ 2	$\triangle \to 2$
④$\to$ 2	$\diamond \to 8$
⑤$\to$ 2	$\pentagon \to 8$
⑥$\to$ 2	$\hexagon \to 4$
⑦$\to$ 2	$\octagon \to 1$
⑧$\to$ 2	$\star \to 8$
♈$\to$ 4	
♉$\to$ 1	
♊$\to$ 4	
♋$\to$ 4	
♌$\to$ 2	
♍$\to$ 4	
♎$\to$ 4	
♏$\to$ 4	

(d) Are any of these groups isomorphic to one another? Explain your reasoning.

Solution: No, these are all unique groups and thus none are isomorphic to one another. One way you can tell is by the order of the elements. The number of elements of a specific order are different for each group.

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A Group and its Characters

4. (20 points) In this problem you will generate a group and compute its character table.

(a) Generate a group using the following 2D mirror and rotation.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{Mirror across } y = \sigma_y$$

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{Counterclockwise rotation by } \frac{\pi}{2} = C_4^{(1)}$$

Express all elements as 2x2 matrices and label each operations using the following notation e for the identity, C_2 for in-plane two-fold rotations, $C_4^{(j)}$ for in-plane counterclockwise four-fold rotations where j is 1 or 3 (Note, $C_2 = C_4^{(2)}$, i for 2D inversion, $\sigma_{x=y}$ for a mirror across the line $x = y$, $\sigma_{x=-y}$ for a mirror across the line $x = -y$, σ_x for a mirror across the line x, and σ_y for a mirror across the line y.

Solution:

$$\underset{e}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \quad \underset{\sigma_x}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \quad \underset{\sigma_{x=y}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \quad \underset{C_4^{(3)}}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$\underset{C_4^{(1)}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \quad \underset{\sigma_{x=-y}}{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}} \quad \underset{\sigma_y}{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} \quad \underset{i = C_2}{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}$$

(b) This group has 5 conjugacy classes. Give the sets of elements in each conjugacy class. For example if both $C_4^{(1)}$ and $C_4^{(3)}$ form a conjugacy class, then the conjugacy class is the set $\{C_4^{(1)}, C_4^{(3)}\}$.

Solution: $\{e\}$, $\{C_2 = i\}$, $\{C_4^{(1)}, C_4^{(3)}\}$, $\{\sigma_{x=y}, \sigma_{x=-y}\}$, $\{\sigma_x, \sigma_y\}$

(c) Compute the trace of the 2D rotation and mirror representations you generated in Part (a) for each conjugacy class. This representation is irreducible. Label which trace belongs to which class.

Solution: $\{e\}$: 2, $\{C_4^{(1)}, C_4^{(3)}\}$: 0, $\{C_2 = i\}$: -2, $\{\sigma_{x=y}, \sigma_{x=-y}\}$: 0, $\{\sigma_x, \sigma_y\}$:0

(d) Using the constraint $\sum_j l_j^2 = h$ where l_j is the dimension of the j^{th} irrep and h is the order (size) of the group, determine the dimensions of the irreps of this group.

Solution: $h = 8$ and we always have the trivial irrep of $l_e = 1$, so dimensions $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$.

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(e) Complete the character table below. Γ_1 is the trivial representation and Γ_2 is the irrep that transforms as your 2D representation of 2D mirrors and rotations from part (a).

- Label the conjugacy classes as the sets of elements in the conjugacy class. For example, label the conjugacy class of $C_4^{(1)}$ and $C_4^{(3)}$ as $\{C_4^{(1)}, C_4^{(3)}\}$. The ordering of the classes does not matter, but using the same order as the instructions for part (a).
- Label the missing irreps as Γ_1', Γ_1'', ... for other 1D irreps and Γ_2', Γ_2'' ... for 2D irreps, as needed. You do not need to worry about absolute ordering, only that the irreps are labeled by the correct dimension.
- Give the characters for the missing irreps using the Wonderful Orthogonality Theorem for Character and what you know about the characters for Γ_1 and Γ_2.

Solution:

	e	$\{C_4^{(1)}, C_4^{(3)}\}$	$\{C_2 = i\}$	$\{\sigma_{x=y}, \sigma_{x=-y}\}$	$\{\sigma_x, \sigma_y\}$
Γ_1	1	1	1	1	1
$\Gamma_{1'}$	1	1	1	-1	-1
$\Gamma_{1''}$	1	-1	1	1	-1
$\Gamma_{1'''}$	1	-1	1	-1	1
$\Gamma_{2'}$	2	0	-2	0	0

Interpreting Outputs

5. (20 points) In the following questions, we will present you code snippets using the functions you have coded in the exercises and ask you to interpret what the outputs means. You may assume the following has been imported.

```
import numpy as np
from symm4ml import groups, group_conv, linalg, rep, vis
import torch
```

(a) Try running the following, but it's taking a long time to evaluate.

```
groups.generate_group(
    np.array([
        [ 0.99994517, -0.01047178 ],
        [ 0.01047178,  0.99994517 ]
    ]).reshape(1, 2, 2)
)
```

Why is this code taking long to evaluate? Explain your reasoning.

> **Solution:** You are generating a very large group, and the code does not scale well with for large groups.

(b) You run the following code snippet and it returns the following output.

```
p3_matrices = groups.permutation_matrices(3)
linalg.infer_change_of_basis(p3_matrices, p3_matrices)
>> array([[[ 5.77350269e-01,  0.00000000e+00,  0.00000000e+00],
        [ 0.00000000e+00,  5.77350269e-01,  0.00000000e+00],
        [ 0.00000000e+00,  0.00000000e+00,  5.77350269e-01]],

       [[-3.70074342e-17,  4.08248290e-01,  4.08248290e-01],
        [ 4.08248290e-01,  7.40148683e-17,  4.08248290e-01],
        [ 4.08248290e-01,  4.08248290e-01, -3.70074342e-17]]])
```

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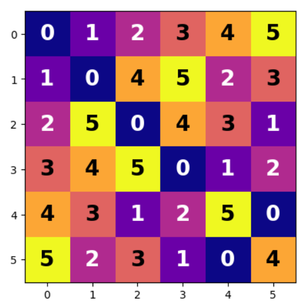
i. Explain what is happening in line (2).

> **Solution:** This function tests whether there exists a matrix M such that Mp3_matricesp3_matric

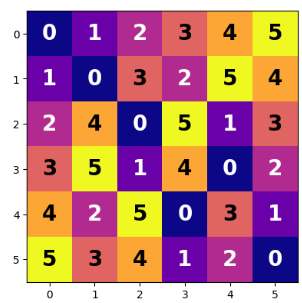
ii. What does the output produced by line (2) tell us about the representation of the 3D permutation matrices? Explain your reasoning.

> **Solution:** By Schur's Lemma (Part 1), if a non-constant matrix commutes with a representation, we know it is reducible. Thus, we know that the representation of the 3D permutation matrices are reducible.

(c) Suppose we have the following multiplication tables saved into `table_dresselhaus` and `table_perm`, respectively.



D_3 Mult. Table



$P(3)$ Mult. Table

What does the following output tell us about these two multiplication tables?

```
groups.isomorphisms(table_dresselhaus, table_perm)
>> {(0, 1, 2, 5, 3, 4),
 (0, 1, 5, 2, 4, 3),
 (0, 2, 1, 5, 4, 3),
 (0, 2, 5, 1, 3, 4),
 (0, 5, 1, 2, 3, 4),
 (0, 5, 2, 1, 4, 3)}
```

Explain what the output means in the context of the rows and columns of the multiplication tables above.

> **Solution:** The groups are isomorphic meaning they are in fact the same group and the multiplication tables can have it's rows and columns permuted to be identical.

(d) In the following snippet, we perform the eigenvalue decomposition of M.

```
M = np.array([[[1, 0, 0], [0, 2, 0], [0, 0, 2]]])
np.linalg.eigh(M)
>> EighResult(
    eigenvalues=array([[1., 2., 2.]]),
    eigenvectors=array([[[1., 0., 0.],
        [0., 1., 0.],
        [0., 0., 1.]]]))
```

Where the eigenvalues are `eigenvalues` and the eigenvectors are given as the columns of `eigenvectors`. Use the re-express M as a sum of projector matrices created using the eigenvectors multiplied by the appropriate eigenvalue. Simplify your expression so that you only have one projector per unique eigenvalue.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 1 \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$