# 6.S966: Exam 2, Spring 2024

# Solutions

- This is a closed book exam. One page (8 1/2 in. by 11 in) of notes, front and back, are permitted. Calculators are not permitted.
- The total exam time is 1 hours and 20 minutes.
- The problems are not necessarily in any order of difficulty.
- Record all your answers in the places provided. If you run out of room for an answer, continue on a blank page and mark it clearly.
- If a question seems vague or under-specified to you, make an assumption, write it down, and solve the problem given your assumption.
- If you absolutely *have* to ask a question, come to the front.
- **Write your name on every piece of paper.**

Name: ______________________ MIT Email: ______________________

| Question | Points | Score |
|---|---|---|
| 1 | 50 | |
| 2 | 35 | |
| 3 | 15 | |
| Total: | 100 | |

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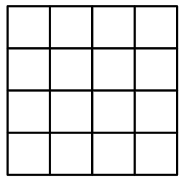
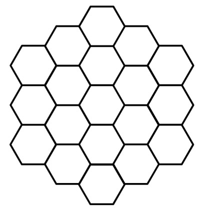
# Steerable Convolution on Hexagonal Images

1. (50 points) There are two regular lattices that tile the 2D plane, square lattices (left) and hexagonal lattices (right).





While it is more common to do convolutions over images made of square pixels, you can also do the same for hexagonal pixels. In this problem, we will determine a basis of hexagonal filters that transform as irreps, parameterize these filters using weights, and perform steerable convolution using tensor product decomposition.

If we neglect translations and center on a specific hexagon pixel, a hexagonal lattice has a point group symmetry of $D_6$ with six-fold (60 degree) rotations and mirrors across the edges and diagonals of the hexagon. The character table for $D_6$ is

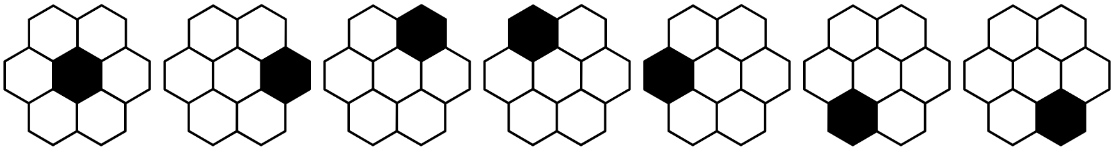
| $\mathbf{D_6}$ | E | $2C_6$ | $2C_3$ | $C_2$ | $3\sigma$ | $3\sigma'$ |
|---|---|---|---|---|---|---|
| $A_1$ | +1 | +1 | +1 | +1 | +1 | +1 |
| $A_2$ | +1 | +1 | +1 | +1 | -1 | -1 |
| $B_1$ | +1 | -1 | +1 | -1 | +1 | -1 |
| $B_2$ | +1 | -1 | +1 | -1 | -1 | +1 |
| $E_1$ | +2 | +1 | -1 | -2 | 0 | 0 |
| $E_2$ | +2 | -1 | -1 | +2 | 0 | 0 |

where the conjugacy classes are the columns and irreps are rows. For the conjugacy classes, $E$ is the identity, $C_n$ are rotations of $2\pi/n$, and $\sigma$ are mirrors.

(a) Use the character table above to determine how many elements are in the point group $D_6$. What parts of the character table tell us how many elements a group has?

**Solution:** The hexagonal lattice has the same symmetry as a hexagon. It consists of the identity, 6-fold rotations (of which there are 5, split into 3-fold and 6-fold conjugacy classes), and mirrors across pairs of edges (3) and diagonals (3). Thus, in total, it has 12 elements. We can see that there are 12 elements in total by looking at the multiplicity of the numbers multiplying the symbol of the conjugacy classes ($1 + 2 + 2 + 1 + 3 + 3 = 12$). Alternatively, we can use the relationship $\sum_i l^i = |G|$ and the first column of the table for $E$ to see that $1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2 = 12$.

(b) A hexagonal filter up to 1st nearest neighbors (1NN) pixels is built from 7 hexagonal pixels. The single pixel basis and corresponding pixel coordinates for the hexagonal 1NN filter are



$(0,0)$ $(1,0)$ $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $(-1,0)$ $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Below, we provide the permutation matrices (grey= 0, black=1) that represent how $D_6$ acts on the single pixel basis (assume the columns are in the same order as the pixel basis).

Below each permutation matrix, fill in the $D_6$ conjugacy class that matrix belongs to. You may assume $3\sigma'$ is the conjugacy class of mirrors that pass through two edges of the central hexagon (leaves three pixels invariant), while $3\sigma$ is the conjugacy class of mirrors through pairs of vertices of the central hexagon.

E 3σ' 3σ $2C^6$ $2C^6$ 3σ

**Solution:** 3σ' $2C^3$ $2C^3$ 3σ' 3σ $C^2$

(c) We define `perm_matrices` as the permutation matrices from above and execute the following code.

```
1 D6_table = groups.make_multiplication_table(perm_matrices)
2 D6_irreps = rep.infer_irreps(D6_table)
3 for i, ir in enumerate(D6_irreps):
4     print(i, linalg.infer_change_of_basis(ir, perm_matrices).shape)
5 > 0 (2, 1, 7)
6 > 1 (1, 1, 7)
7 > 2 (0, 1, 7)
8 > 3 (0, 1, 7)
9 > 4 (1, 2, 7)
10 > 5 (1, 2, 7)
```

i. Describe what's happening in lines 1 and 2. In particular, how does `rep.infer_irreps` use its input to obtain irreps of the group?

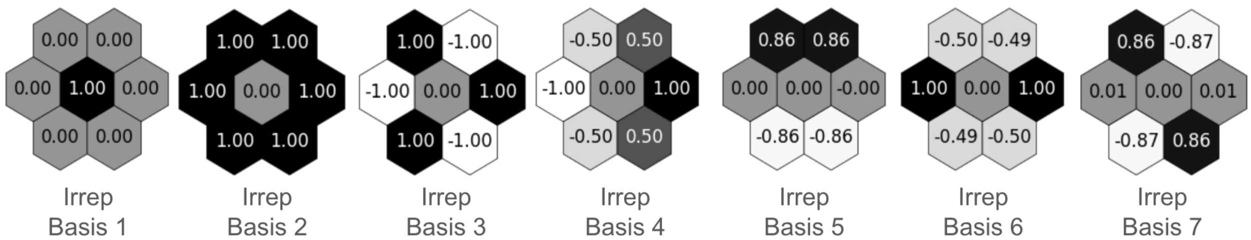
> **Solution:** `rep.infer_irreps` creates the left regular representation from the multiplication table and feed that to `rep.decompose_rep_into_irreps`. It then checks for isomorphic irreps and only returns those that are unique.

ii. Describe what is happening in lines 3-4. In particular, what is the significance of the shape of the output of `linalg.infer_change_of_basis`? How many irreps are contained in the representation `perm_matrices`?

> **Solution:** For each iteration, we are seeing if there exists a change of basis between one of the irreps of $D_6$ and the representation `perm_matrices`. By looking at the zeroth index of the shape for each iteration, we see that there are $2 + 1 + 1 + 1 = 5$ irreps contained in `perm_matrices` that span the 7 dimensions. By looking at the first index of the shape, two of the irreps are 2D while the rest are 1D).

(d) The outputs of `linalg.infer_change_of_basis` give us the change of basis between the single pixel basis and specific irreps. This means the change of basis gives coefficients indicating how much of each single pixel basis is contained in each irrep basis. We can plot these coefficients to visualize the 7 irrep basis functions below. These basis functions transform as a direct sum the following irreps of $D_6$:

$$\rho_{\text{Hex 1NN}} = 2A_1 \oplus B_2 \oplus E_1 \oplus E_2 \tag{1}$$




i. Irrep Basis 1, 2, and 3 correspond to the output from part (c) for $i = 0$ and $i = 1$. Using the $D_6$ character table, determine which of these basis functions transform as $A_1$ vs. $B_2$. Explain your reasoning.

> **Solution:** Irrep Basis 1 and 2 transform as the irrep $A_1$ because they are invariant under group action. Irrep Basis 3 transforms as the irrep $B_2$ because it is invariant under mirrors across two vertices (i.e. $3\sigma'$), invariant under $2C_3$ rotations, but not invariant under $2C_6$.

ii. Given your answers in part (i), explain why there are two basis functions that transform as $A_1$? How are they similar? How are they different?

> **Solution:** There are two such functions because there are two distinct radii – the zero pixel and the shell of 1st nearest neighbors.

iii. Irrep Basis 4 and 5 correspond to the output from part (c) for $i = 4$. Using the $D_6$ character table, determine which 2D irrep these basis functions transforms as. Explain your reasoning.

> **Solution:** These basis functions transform as the irrep $E_1$ because they correspond to a 2D irrep but are not invariant under $C_2$.

iv. Irrep Basis 6 and 7 correspond to the output from part (c) for $i = 5$. Using the $D_6$ character table, determine which 2D irrep these basis functions transforms as. Explain your reasoning.

> **Solution:** These basis functions transform as the irrep $E_2$ because they correspond to a 2D irrep and are invariant under $C_2$.

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(e) Now, we want to add weights to our basis functions to parameterize convolutional kernels, $\psi(x, y) = WB(x, y)$. The weights matrix $W$ for our kernel $\psi$ is a linear map $W : \rho_{\text{basis}} \to \rho_{\text{filter}}$ and thus has shape $\rho_{\text{filter}} \times \rho_{\text{basis}}$, i.e. the rows span $\rho_{\text{filter}}$ and the columns span $\rho_{\text{basis}}$.

To commute with group action $W$ must satisfy the following,

$$W^{\rho_{\text{filter}} \times \rho_{\text{basis}}} D^{\rho_{\text{basis}}}(g) x^{\rho_{\text{basis}}} = D^{\rho_{\text{filter}}}(g) W^{\rho_{\text{filter}} \times \rho_{\text{basis}}} x^{\rho_{\text{basis}}}. \tag{2}$$

where $D^{\rho}$ is the matrix representation for representation vector space $\rho$ and is therefore a $\rho \times \rho$ matrix.

Assume $\rho_{\text{basis}} = \rho_{\text{filter}} = \rho_{\text{Hex 1NN}} = 2A_1 \oplus B_2 \oplus E_1 \oplus E_2$ (in that order). Below, fill in the weight matrix $W$ such that it commutes with group action. Use lower case Latin letters $(a, b, \ldots, z)$ to label distinct weights. You may leave entries blank or use zeros to indicate zeros.

$\rho_{\text{basis}} \rightarrow$

$\rho_{\text{filter}} \downarrow$

| | | | | | | |
|---|---|---|---|---|---|---|
| a | b | | | | | |
| c | d | | | | | |
| | | e | | | | |
| | | | f | | | |
| | | | | f | | |
| | | | | | g | |
| | | | | | | g |

**Solution:**

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(f) To perform steerable convolution, we perform an elementwise tensor product of our filter and the image patch the filter overlaps. The direct product table for the irreps of $D_6$ is given below. For example, using the table we see $E_1 \otimes B_2 = E_2$.

| | **A₁** | **A₂** | **B₁** | **B₂** | **E₁** | **E₂** |
|---|---|---|---|---|---|---|
| **$A_1$** | $A_1$ | $A_2$ | $B_1$ | $B_2$ | $E_1$ | $E_2$ |
| **$A_2$** | $A_2$ | $A_1$ | $B_2$ | $B_1$ | $E_1$ | $E_2$ |
| **$B_1$** | $B_1$ | $B_2$ | $A_1$ | $A_2$ | $E_2$ | $E_1$ |
| **$B_2$** | $B_2$ | $B_1$ | $A_2$ | $A_1$ | $E_2$ | $E_1$ |
| **$E_1$** | $E_1$ | $E_1$ | $E_2$ | $E_2$ | $A_1 \oplus [A_2] \oplus E_2$ | $B_1 \oplus B_2 \oplus E_1$ |
| **$E_2$** | $E_2$ | $E_2$ | $E_1$ | $E_1$ | $B_1 \oplus B_2 \oplus E_1$ | $A_1 \oplus [A_2] \oplus E_2$ |

If the input is a hexagonal image with features that transform as $A_2$, and we use $\rho_{\text{filter}} = 2A_1 \oplus B_2 \oplus E_1 \oplus E_2$, how will the outputs of tensor product transform? In other words, how does $A_2 \otimes (2A_1 \oplus B_2 \oplus E_1 \oplus E_2)$ decompose into irreps?

**Solution:**

$$A_2 \otimes (2A_1 \oplus B_2 \oplus E_1 \oplus E_2) = 2A_2 \oplus B_1 \oplus E_1 \oplus E_2 \tag{3}$$

* [NOT REQUIRED FOR EXAM] To expand on the above, we center our filter on each pixel $i$ of our input $U$ and compute the following

$$V_i^{\rho_{\text{out}}} = \sum_{j \in N(i)} U_j^{\rho_{\text{in}}} \otimes \psi^{\rho_{\text{filter}}}(\{x_{ij}, y_{ij}\}) = \sum_{j \in N(i)} Q^{\rho_{\text{out}}}_{\rho_{\text{in}}, \rho_{\text{filter}}} U_j^{\rho_{\text{in}}} \psi^{\rho_{\text{filter}}}(\{x_{ij}, y_{ij}\}) \tag{4}$$

where $j \in N(i)$ indicates the pixels that are overlapped by the filter when the filter is centered on $i$, $Q^{\rho_{\text{out}}}_{\rho_{\text{in}}, \rho_{\text{filter}}}$ is the change of basis from $\rho_{\text{in}} \otimes \rho_{\text{filter}} \to \rho_{\text{out}}$, $\{x_{ij}, y_{ij}\}$ indicates the relative distances between $i$ and $j$ used to get the filter value, and $V$ is the output image.

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## Planes of 4D Rotations

2. (35 points) In this question, we will investigate the Lie group of 4D rotations $SO(4)$.

   (a) Representations of Lie groups take the form of $e^{\sum_i \theta_i X_i}$ with parameters $\theta_i$ multiplying generators $X_i$, where $e^A = \sum_k \frac{1}{k!}A^k$. Like finite groups, Lie groups are closed under group multiplication. If $X$ and $Y$ are matrices that do not necessarily commute, how do we compute $Z$ (or an approximation of $Z$) in $e^X e^Y = e^Z$? Explain your reasoning.

   > **Solution:** If $X$ and $Y$ are matrices that do not commute, $e^X e^Y \neq e^{X+Y}$. To evaluate what this is equivalent to, we need to use the Campbell-Baker-Hausdorff identity, i.e. $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{3!}[X,[X,Y]]+\frac{1}{3!}[Y,[X,Y]]\dots}$.

   (b) Orthogonal matrices have the property $R^T R = RR^T = I$ which means $R^T = R^{-1}$. Suppose we have an orthogonal matrix generated by $A$, i.e., $R = e^A$. We can use the definition of $e^A = \sum_k \frac{1}{k!}A^k$ to see $R^T = e^{A^T}$. What condition do we have on $A^T$ if $e^A e^{A^T} = I = e^0$? What must the diagonals of these generators be?

   > **Solution:** $A^T = -A$ which means $A$ are skew-symmetric matrices. The diagonals must be zero.

   (c) There are 3 generators for $SO(3)$ and 6 generators for $SO(4)$. What would you expect the number of generators to be for $SO(5)$? You can derive this using the condition above for $5 \times 5$ matrices. Alternatively, you may use the fact that in $n$-dimensional space, there are $n$ choose 2 (i.e., $\frac{n!}{2!(n-2)!}$) planes but in that case explain how are rotations are connected to planes. In case helpful, the generators for $SO(4)$ are given on the next page.

   > **Solution:** $5 \times 5$ skew symmetry matrices have equal and opposite off diagonals and zeros on the diagonals so $(25-5)/2 = 10$.
   >
   > Alternatively, rotations are defined by planes. In 3D, the number of directions vs. planes is the same (i.e. we get lucky). But in 4D it becomes clear that the number of planes determines the number of distinct rotations (6 planes leads to 6 generators). For 5D, $\frac{5!}{2!3!} = \frac{5\dot{4}}{2} = 10$

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(d) Given the condition above, we can determine the generators for $SO(n)$ of any $n$. The six generators of $SO(4)$ can be written as:

$$L_1 = L_{(01)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_2 = L_{(02)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_3 = L_{(03)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$L_4 = L_{(12)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_5 = L_{(13)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad L_6 = L_{(23)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

where $L_i$ enumerates the 6 generators and $L_{(jk)}$ specifies the plane of rotation.

i. Compute the commutators $[L_{(01)}, L_{(12)}]$ and $[L_{(12)}, L_{(13)}]$

**Solution:**

$$[L_{(01)}, L_{(12)}] = L_{(02)} \tag{5}$$
$$[L_{(12)}, L_{(13)}] = -L_{(23)} \tag{6}$$

ii. Compute the commutators of $[L_{(01)}, L_{(23)}]$ and $[L_{(02)}, L_{(13)}]$.

**Solution:**

$$[L_{(01)}, L_{(23)}] = 0 \tag{7}$$
$$[L_{(02)}, L_{(13)}] = 0 \tag{8}$$

(e) Given what you computed above, describe in words the following cases for the Lie algebra (commutator relationship) of $SO(4)$: Under what circumstances are the commutators zeros? Under what circumstances are the commutators non-zero? Don't forget to handle the case of the $[L_{(ij)}, L_{(ij)}]$. From a geometric perspective (thinking of how rotations are related to planes), does this make sense?

> **Solution:** The commutators are only nonzero when the two generators share one direction that define the rotation planes. If they share both or no directions, the commutator is zero. In the case of one shared direction, the resulting commutator gives the generator of the two unshared directions, i.e. $[L_{(01)}, L_{(12)}] = L_{(02)}$.
>
> This makes sense geometrically because it means rotations only don't commute if they "mix" similar directions. If they mix independent directions, they can be applied in any order.
>
> [Not required] Because of the specific way we have our minus signs in the generators, the commutator has a plus sign if the shared directions are on the "outside" or "inside", i.e. $[L_{(23)}, L_{(12)}]$ or $[L_{(01)}, L_{(12)}]$. Otherwise, the commutator has a minus sign. Not required but for the sake of completeness – we can put this all together to write,
>
> $$[L_{(mn)}, L_{(qp)}] = \delta_{(mp)} L_{(nq)} + \delta_{(nq)} L_{(mp)} - \delta_{(mq)} L_{(np)} - \delta_{(np)} L_{(mq)}$$

(f) We execute the following code.

```
1 so4_vec_vec = lie.tensor_product(so4_generators, so4_generators)
2 np.random.seed(42)
3 so3_vec_vec_irreps = lie.decompose_rep_into_irreps(so4_vec_vec)
```

In line 2, we are setting a random seed which makes the use of random numbers reproducible. How is randomness used in `lie.decompose_rep_into_irreps` to arrive at the irreps contained in the tensor product representation `so4_vec_vec`? Feel free to use an `einsum` to help give your explanation.

> **Solution:** Randomness is used create a number of random coefficients that dot product with the solutions from `linalg.infer_change_of_basis(rep, rep)` to help create a matrix with non-degenerate eigenspaces, e.g. if
> `Qs = linalg.infer_change_of_basis(rep, rep)` outputs a solution `Qs.shape` is `(m, n, n)` we use `m` random coefficients
> `new_Q = np.einsum('m,mnp->np', rand_coeff, Qs)`. We then perform an eigenvalue decomposition of `new_Q` to find its eigenspaces which due to the randomness should be vector spaces of irrep.

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## Spherical Harmonic Identity

3. (15 points) From elementary trigonometry, we know that $\sin(\theta)^2+\cos(\theta)^2 = 1$ for any $\theta$. In this question, we prove a generalized version of this identity for spherical harmonics. Let $Y_{\ell,m}(\xi)$ denote the spherical harmonics for $\ell = 0, 1, \ldots$ and $\ell \leq m \leq \ell$. Here, $\xi = (x, y, z)^T \in \mathbb{R}^3$ is a vector. Also, assume that we normalized spherical harmonics such that

$$\int_{S^2} \left|Y_{\ell,m}(\xi)\right|^2 d\xi = 1 \tag{9}$$

for each $\ell, m$, where the integral is over the sphere $S^2 = \left\{\xi \in \mathbb{R}^3 : |\xi|^2 = 1\right\}$.

In this problem, we want to show that for any $\ell$ and any $\xi \in S^2$,

$$\sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\xi)\right|^2 = \frac{2\ell+1}{4\pi}. \tag{10}$$

(a) First, define the following function:

$$f(\xi_1, \xi_2) = \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\xi_1) Y_{\ell,m}(\xi_2). \tag{11}$$

Use the properties of spherical harmonics under 3D rotation

$$Y_{\ell,m}(D^\xi(g)\xi) = \sum_{m'=-\ell}^{\ell} D^\ell_{m,m'} Y_{\ell,m'}(\xi) \tag{12}$$

and the property of orthogonal matrices $D^\ell_{ij}(g) = D^\ell_{ji}(g^{-1})$ to show that $f(\xi_1, \xi_2)$ is invariant under rotation, i.e. $f(D^\xi(g)\xi_1, D^\xi(g)\xi_2) = f(\xi_1, \xi_2)$.

**Solution:** Note that for any $g \in SO(3)$, we have

$$\begin{aligned} f(D^\xi(g)\xi_1, D^\xi(g)\xi_2) &= \sum_{m=-\ell}^{\ell} Y_{\ell,m}(D^\xi(g)\xi_1) Y_{\ell,m}(D^\xi(g)\xi_2) \\ &= \sum_{m=-\ell}^{\ell} \left( \sum_{m'=-\ell}^{\ell} D^\ell_{m,m'}(g) Y_{\ell,m'}(\xi_1) \sum_{m''=-\ell}^{\ell} D^\ell_{m,m''}(g) Y_{\ell,m''}(\xi_2) \right) \\ &= \sum_{m'=-\ell}^{\ell} \sum_{m''=-\ell}^{\ell} Y_{\ell,m'}(\xi_1) Y_{\ell,m''}(\xi_2) \sum_{m=-\ell}^{\ell} D^\ell_{m,m'}(g) D^\ell_{m,m''}(g). \end{aligned}$$

Now let us compute $\sum_{m=-\ell}^{\ell} D^\ell_{m,m'}(g) D^\ell_{m,m''}(g)$. Note that using the properties of orthogonal matrices, we have

$$\sum_{m=-\ell}^{\ell} D^\ell_{m,m'}(g) D^\ell_{m,m''}(g) = \sum_{m=-\ell}^{\ell} D^\ell_{m',m}(g^{-1}) D^\ell_{m,m''}(g) \tag{13}$$

$$= \sum_{m=-\ell}^{\ell} D^\ell_{m',m''}(g^{-1}g). \tag{14}$$

But $g^{-1}g = e$ where $e_G$ is the identity element in the group $SO(3)$, so $D^\ell_{m',m''}(g^{-1}g) = D^\ell_{m',m''}(e)$. The matrix $D^\ell(e)$ is the identity matrix of dimension $(2\ell+1)$. Thus, $D^\ell_{m',m''}(g^{-1}g) = \delta_{m',m''}$, which means that it is one if $m' = m''$ and zero otherwise. Thus, we conclude that

$$\begin{aligned} f(D^\xi(g)\xi_1, D^\xi(g)\xi_2) &= \sum_{m'=-\ell}^{\ell} \sum_{m''=-\ell}^{\ell} Y_{\ell,m'}(\xi_1) Y_{\ell,m''}(\xi_2) \delta_{m',m''} \\ &= \sum_{m'=-\ell}^{\ell} Y_{\ell,m'}(\xi_1) Y_{\ell,m'}(\xi_2) \\ &= f(\xi_1, \xi_2). \end{aligned}$$

In particular, $f(\xi_1, \xi_2)$ is invariant under the action of $SO(3)$ on both vectors (simultaneously). This means that the function $f(\xi_1, \xi_2)$ only depends on the inner product of the two vectors, and this completes the proof.

(b) Note, that because $f(\xi_1, \xi_2)$ is invariant for any $\xi_1$ and $\xi_2$, for $\xi_1 = \xi_2 = \xi$, $f(\xi, \xi) = \sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\xi)\right|^2$ is still invariant. Prove that $\sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\xi)\right|^2 = \frac{2\ell+1}{4\pi}$.
*Hint.* Integrate the left-hand side of the above identity. You may also find Eqn. 9 helpful. Also, $\int_{S^2} d\xi = 4\pi$.

**Solution:** We integrate the LHS of the above with respect to $\xi$ over the sphere $S^2$:

$$\int_{S^2} \sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\xi)\right|^2 d\xi = \sum_{m=-\ell}^{\ell} \int_{S^2} \left|Y_{\ell,m}(\xi)\right|^2 d\xi \tag{15}$$

$$= \sum_{m=-\ell}^{\ell} 1 = 2\ell + 1. \tag{16}$$

But note that we also have

$$\int_{S^2} \sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\xi)\right|^2 d\xi = \sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\xi)\right|^2 \int_{S^2} d\xi = 4\pi \sum_{m=-\ell}^{\ell} \left|Y_{\ell,m}(\xi)\right|^2, \tag{17}$$

because the integrand is a constant (i.e., it does not depend on $\xi$). This completes the proof.