Representation Reduction

Mario Geiger and Hannah Lawrence

February 2023

1 Introduction

Given a vector space V and a representation $\rho : G \to GL(V)$, we seek to decompose it into irreps:

$$\rho(g) = U \bigoplus_{i=1}^{r} \rho_i(g)\ U^{-1}\ \ \forall g \in G \tag{1}$$

Here, U is a change of basis matrix. We will use ρ_i to denote an irreducible representation acting on a vector space of dimension d_i, i.e. $\rho_i : G \to \mathbb{C}^{d_i \times d_i}$. Concretely, we will output irreducible representations $\rho'_1, \ldots, \rho'_r$ such that a U satisfying eq. (1) exists.

When working with representations, we won't always write the group element g; assume equalities in terms of representations hold for all $g \in G$ if not specified!

2 Algorithm

1. First, obtain a random matrix Q that satisfies,

$$Q\rho(g) = \rho(g)Q. \tag{2}$$

   More precisely, if $\{Q_i, \ldots, Q_r\}$ form a basis for the solution space of matrices satisfying eq. (2), draw $\alpha_i \sim U([0, 1])$ and let $Q = \sum_{i=1}^{r} \alpha_i Q_i$

2. Diagonalize Q to obtain its eigenspaces (λ_i, W_i), where each λ_i is an eigenvalue and W_i is the corresponding eigenspace. Note that the eigenspaces W_i will not all be orthogonal subspaces to one another (typically if an irrep with multiplicity greater than 1 appears in ρ), so we cannot necessarily say that $V = \bigoplus_i W_i$.

3. Nonetheless, it turns out that the eigenspaces W_i are minimal stable (invariant) subspaces of ρ. As a result, ρ restricted to W_i is equal to ρ_i! Concretely, if the projector P_i corresponding to W_i is equal to $B_i B_i^\dagger$ where B_i has orthonormal columns, then we can set $\rho_i = B_i^\dagger \rho B_i$.

3 Proof Outline

We sketch the key ideas of the proof that this algorithm works below.

3.1 Eigenspaces are stable spaces

We say that a vector space V is a **stable** space of a representation ρ if $\rho(g)(V) \subseteq V$ for all $g \in G$ ($\Rightarrow \rho(g)(V) = V$ because ρ is a representation, as is explained in eq. (3) below). This is also sometimes called a **G-invariant** subspace of ρ. Here, we use the notation $\rho(g)(V)$ to denote the vector space $\{\rho(g)v : v \in V\}$.

The vector spaces corresponding to irreducible representations can be characterized as stable, or invariant, subspaces. More precisely, a representation ρ acting on vector space V has no stable subspaces other than V and $\{0\}$ if and only if it is irreducible! If a representation ρ has nontrivial stable subspaces, then the subrepresentations given by restricting ρ to its smallest stable subspaces are the irreducible representations comprising ρ.

We begin by showing that the eigenspaces of Q are stable subspaces. We'll then argue they are minimal (with probability 1) in the next section.

Let W be the eigenspace of Q corresponding to eigenvalue λ. Here Q is any self-similarity transform of ρ, i.e. any matrix satisfying eq. (2) ($Q\rho = \rho Q$). Let $\vec{u} \in W$ be an eigenvector in this eigenspace. We first show that for all g, $\rho(g)(W) \subseteq W$:

$$Q\rho(g)\vec{u} = \rho(g)Q\vec{u} = \rho(g)\lambda\vec{u} = \lambda\rho(g)\vec{u}.$$

As a result, $\rho(g)\vec{u} \in W$, as $\rho(g)\vec{u}$ is in the eigenspace of Q with eigenvalue λ.

Note that $\rho(g)(W) \subseteq W \; \forall g$ implies that $W \subseteq \rho(g)(W) \; \forall g$:

$$\rho(g^{-1})(W) \subseteq W \rightarrow \rho(g)^{-1}(W) \subseteq W \rightarrow W \subseteq \rho(g)(W) \tag{3}$$

Therefore, $\rho(g)(W) = W$ for all g as desired, and W is a stable space of ρ.

3.2 Eigenspaces are *irreducible* stable spaces

As shown in the previous section, the eigenspaces of any Q satisfying eq. (2) are always stable spaces of ρ. But when are they the minimal (or irreducible) stable spaces? This actually depends on the specific Q obtained. But we can argue that if Q is sampled randomly, it is true with probability 1.

3.2.1 $\exists$ good Q's

First, to aid in our understanding, let's just show that there *exists* a Q satisfying eq. (2) that has irreducible stable spaces as its eigenspaces. For this, we can simply construct one: let W_i' be the (minimal) stable space associated with the irrep ρ_i in the decomposition eq. (1). More precisely, let W_i' be the span of the column vectors of U corresponding to the indices of ρ_i in the block diagonal matrix $\bigoplus_{i=1}^{r} \rho_i(g)$. Let P_i the orthogonal projector onto this vector space. Note

that if we assume that U is unitary (for the sake of simplicity of the argument – this is valid if ρ and the irreps are unitary), $V \cong W_1' \oplus \cdots \oplus W_r'$. Since each ρ_i is irreducible, we have that $\rho(W_i') = W_i' \; \forall i$, and $W \subseteq W_i'$ satisfying $\rho(W) = W$ implies that $W = \{0\}$ or $W = V$. So, we can of course write $\rho = \sum_{i=1}^{r} P_i \rho P_i$, using that for all i, $\rho(g) P_i \vec{v} \in W_i' \; \forall \vec{v} \in V$ by the irreducibility of W_i'. Then, setting

$$Q := \sum_i \lambda_i P_i, \;\; \lambda_i \text{ are unique}, \tag{4}$$

we have that Q satisfies (2):

$$Q\rho = \sum_i \lambda_i P_i \rho = \sum_i \lambda_i \rho P_i = \rho Q,$$

where we used that ρ and P_i commute:

$$\rho P_i = \sum_j P_j \rho P_j P_i = P_i \rho P_i = P_i \rho$$

Moreover, so long as the eigenvalues λ_i are unique, its eigenspaces are the irreducible stable subspaces W_i by construction:

$$Q\vec{w}_i = \lambda_i \vec{w}_i \;\; \forall \vec{w}_i \in W_i.$$

3.2.2 They are almost all good

It remains to show that the Q's that have this property are actually dense, and the ones that don't (for instance $Q = I$) are of measure zero. For the sake of simplicity, we will first argue it in the easier special case of two non-equivalent irreps:

$$\rho(g) = U(\rho_1(g) \oplus \rho_2(g))U^{-1} \tag{5}$$

(Recall that two representations ρ and ρ' are equivalent if there is a linear bijection T satisfying $T\rho = \rho' T$.)

In this case, by Schur's Lemma, any Q satisfying eq. (2) can be written as:

$$Q = \lambda_1 P_1 + \lambda_2 P_2, \tag{6}$$

where P_1 and P_2 are as in the previous section. This is "bad" when $\lambda_1 = \lambda_2$, because then Q only has one eigenspace, instead of two, and the eigendecomposition does not allow us to distinguish between the two minimal stable subspaces of ρ. The space of solutions to the self-similarity transform problem is then isomorphic to $\mathbb{R}^2$, where the domain of degenerate eigenvalues (reducible eigenspaces) corresponds to a line, which is therefore of measure zero under any reasonable distribution over Q's.

Let's now consider a sketch for the more challenging case of irreps with arbitrary multiplicities. By Schur's Lemma, $U^{-1}QU$ takes a particular block-diagonal form, where each block is a Kronecker product between a square matrix with side length equal to the multiplicity of the irrep, and an identity matrix

with side length equal to the dimension of the irrep. We illustrate this in block form below for the illustrative example $\rho_1 \oplus \rho_2 \oplus \rho_2$. Let $U^{-1}QU = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} \rho_1 & 0 & \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_2 \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_2 \end{bmatrix} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} A\rho_1 & B\rho_2 & C\rho_2 \\ D\rho_1 & E\rho_2 & F\rho_2 \\ G\rho_1 & H\rho_2 & I\rho_2 \end{bmatrix} = \begin{bmatrix} \rho_1 A & \rho_1 B & \rho_1 C \\ D\rho_2 & E\rho_2 & F\rho_2 \\ G\rho_2 & H\rho_2 & I\rho_2 \end{bmatrix}$$

$\rightarrow B, C, D, G = 0,\ A = aI, E = eI, F = fI, H = hI, I = iI$ by Schur's Lemma

$$\rightarrow U^{-1}QU = \begin{bmatrix} aI & 0 & 0 \\ 0 & eI & fI \\ 0 & hI & iI \end{bmatrix} = \begin{bmatrix} aI & & 0 \\ & \begin{bmatrix} e & f \\ h & i \end{bmatrix} \otimes I & \\ 0 & & \end{bmatrix}$$

The diagonalization of $U^{-1}QU$ can then be written in terms of the diagonalizations of the square non-identity matrices in the Kronecker products. We assume that with probability 1, (1) these matrices have eigendecompositions and (2) their eigenvalues are all unique. (Recall that the diagonalization of a matrix yields its eigenvectors as columns, even though these eigenvectors may not be mutually orthogonal.)

3.3 Using the irreducible stable spaces to obtain ρ_i's

We can now set $\rho_i = B_i^\dagger \rho B_i$, where the columns of B_i form an orthonormal basis for the irreducible stable space W_i of ρ. The ambiguity in the choice of B_i corresponds to the ambiguity in the choice of basis for ρ_i!

As a simple exercise, we check that so long as B_i describes an irreducible stable space of ρ, then this choice of ρ_i is indeed an irreducible representation. To do this, we proceed by contradiction: suppose there exists a subspace $W \subset W_i$ where $W \neq W_i$, $W \neq \{0\}$ that is invariant under ρ_i, so $\rho_i(B_i^\dagger W) = B_i^\dagger W$. Then, $B_i^\dagger \rho P_i W = B_i^\dagger W$. By left-multiplying by B_i, we obtain $P_i \rho P_i W = P_i W \rightarrow P_i \rho W = W \rightarrow \rho W = W$, which contradicts that W_i is an irreducible stable space of ρ.